The Hermitian Adjacency Spectra of Digraphs with Morita Equivalent C^*-Algebras

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Abstract

A directed graph, or digraph consists of a set of vertices and edges in which each edge has a source and a range within the set of vertices. Each digraph has a number of matrices associated with it, including the Hermitian adjacency matrix. This matrix is indexed by the digraph's vertex set, and it characterizes which vertices are connected by an edge and which direction this edge points. There are six particular moves which can be preformed on a digraph and maintain a specific property of the digraph (the Morita equivalence class of its C*-Algebra). However, none of the moves maintain the Hermitian adjacency spectrum, the multiset of eigenvalues of the Hermitian adjacency matrix. Thus, we are investigating and characterizing how this spectrum changes when we perform a sequence of these moves and their inverses.

Contents

1 Introduction

A directed graph, or digraph consists of a set of vertices and edges in which each edge has a source and a range within the set of vertices. Digraphs can be used to model a variety of different systems: flight schedules, neurons and axons in the brain, charts drawn on high school white boards explaining who has a crush on who, tournament lineups, and more. Virtually any system in which you may want to indicate a directed relationship between two objects can be realized as a digraph.

With a wide variety of digraphs come a variety of matrices. There are numerous ways to associate matrices to digraphs. Perhaps the most intuitive is exhibited in the *adjacency matrix*, which essentially describes which vertices are connected by an edge (or *adjacent*). However, specifically for directed graphs, this matrix is not guaranteed to be symmetric, and therefore may not have real eigenvalues. This problem is solved by the *skew-symmetric adjacency matrix*. But this matrix disregards the existence of digons – a pair of edges which point in opposite directions between to vertices. In contrast, the *Hermitian adjacency matrix* both ensures real eigenvalues and takes digons into account. This matrix – which will be a main focus of our study – is indexed by the digraph's vertex set, and it characterizes which vertices are connected by an edge and which direction this edge points.

In this project, we are interested in performing six specific moves or changes that maintain a specific quality of a digraph and, in turn, analyzing how these moves affect the spectra of the Hermitian adjacency matrix. We already know that none of the moves maintain the Hermitian Adjacency Spectrum, the multiset of eigenvalues of the Hermitian Adjacency Matrix. The purpose of this project is mainly to perform preliminary exploration into the broad question: how does the Hermitian adjacency spectra of a digraph change through our specific digraph moves? Numerous routes have been taken in this exploration, all of which will be explained in the following paper. For instance, we have specifically characterized how applying one move in one specific way will change a digraph's Hermitian adjacency matrix. From this, we were able to find an expression for the digraph's Hermitian adjacency spectrum, provided the original matrix is determined. Additionally, we have looked at questions of spectral radius, asking if we can send the largest eigenvalue (or absolute value of the smallest) to infinity through repeated moves.

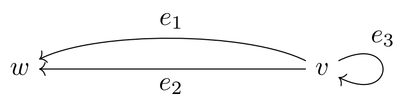
In this paper, we will outline the background necessary for understanding the project and its motivation, walk through the performance of Move $(S)^{-1}$ in detail, outline a process to find the spectrum of a digraph with $(S)^{-1}$ applied numerous times to it, discuss applying $(S)^{-1}$ to different vertices, discuss spectral radius, and briefly look at Move $(R)^{-1}$.

2 Background

2.1 Digraphs

A directed graph, or *digraph*, is comprised of a set of *vertices*, V, and a set of *edges*, E along with two functions, the *source*, $s : E \to V$, and *range*, $r : E \to V$. In other words, each edge, e, points from a vertex, $s(e) = v_1$, to another vertex, $r(e) = v_2$.

In this project, we will allow digraphs to have both parallel edges: two edges, e_1, e_2, such that $s(e_1) = s(e_2)$ and $r(e_1) = r(e_2)$, and loops: an edge, e, such that $s(e) = r(e)$. Illustrated below is a digraph with loop e_3 and parallel edges e_1 and e_2.



There is a wide range of vocabulary used to describe different properties of digraphs. Here, we will give a brief and intuitive outline of the definitions we will need. A more exhaustive list of definitions can be found in [3].

1. *path*: a finite sequence of edges (such that the range of previous edge is the source of next edge) between to specific vertices. A path's *length* is the number of edges in this sequence.
2. *cycle*: a path which begins and ends at the same vertex. A cycle is *simple* if it repeats no edges. It is *vertex simple* if it repeats no vertices.
3. *sink*: a vertex which has no edges pointing away from it
4. *source*: a vertex which has no edges pointing into it
5. *regular vertex*: a vertex with at least one edge pointing towards it
6. *indegree*: the number of edges which point into a specific vertex
7. *outdegree*: the number of edges which point out of a specific vertex
8. *digon*: a pair of edges, $e_1, e_2 \in E$ such that $s(e_1) = r(e_2)$ and $s(e_2) = r(e_1)$
9. *underlying digraph*: the graph obtained by making every edge undirected and then removing any parallel edges
10. *underlying degree of a vertex*: the number of edges a single vertex touches in a digraph's underlying graph
11. *maximum degree of an underlying graph*: the largest degree of a single vertex in the underlying graph of a digraph

2.2 The Hermitian adjacency matrix and spectrum

The Hermitian adjacency matrix is one of many matrices associated to a digraph. As indicated by it name, this matrix is Hermitian, meaning it is equal to its Hermitian transpose, or the complex conjugate of its transpose. Because of this, the Hermitian adjacency matrix has many nice properties, the most influential to us being that it will always have real eigenvalues.

Definition 2.1. The *Hermitian adjacency matrix* of a digraph D, $\mathbf{H}(D) = (h_{v,w})_{v,w \in V}$, is the square matrix whose rows and columns are indexed by V such that:

$$h_{vw} = \begin{cases} +1, & s^{-1}(v) \cap r^{-1}(w) \neq \emptyset \text{ and } s^{-1}(w) \cap r^{-1}(v) \neq \emptyset \\ +i, & s^{-1}(v) \cap r^{-1}(w) \neq \emptyset \text{ and } s^{-1}(w) \cap r^{-1}(v) = \emptyset \\ -i, & s^{-1}(v) \cap r^{-1}(w) = \emptyset \text{ and } s^{-1}(w) \cap r^{-1}(v) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

Definition 2.2. The *Hermitian adjacency spectrum*, of digraph D, $Spec_{\mathbf{H}}(D)$, is the multiset of eigenvalues of the Hermitian adjacency matrix of D. The *spectral radius* of $Spec_{\mathbf{H}}(D)$ is the largest element of the abolute values of this set.

2.3 C*-Algebras and Digraph Moves

Each digraph is associated with a $C*$-algebra, which is an object of interest in many fields of pure mathematics. However, their precise definition is not necessary for our discussion. Each of these $C*$-algebras fit into a Morita equivalence class. A recent paper by Eilers et. al. [2], outlined six specific moves (and their inverses) which, when preformed on a digraph, maintain the Morita equivalence class of its associated $C*$-algebra. For our purposes, we need only think of these as six moves which maintain an important quality of our digraph.

These six moves are:

- *Move (S): remove a regular source*
- *Move (R): reduce at a regular vertex*
- *Move (O): outsplit at a non-sink*
- *Move (I): in-split at a regular non-source*
- *Move (C): the Cuntz splice at a vertex with at least two return paths*
- *Move (P): enclose a cyclic component*

The two moves of most interest to us are S^{-1} and R^{-1}, defined below. The rest are thoroughly defined in [2].

Definition 2.3. *Move S^{-1}: add a regular source*
Let $D = (V, E, r, s)$ be a digraph. Let w be a vertex not in D. Let $\{v_1, ..., v_n\}$ be any subset of V. Then, $D^{(S^{-1})} = (V^{(S^{-1})}, E^{(S^{-1})}, r^{(S^{-1})}, e^{(S^{-1})})$ (D with move S^{-1} applied) is defined as such:

$$V^{(S^{-1})} = V \cup \{w\}; E^{(S^{-1})} = E \cup \{e_1, ..., e_n\};$$

For $e \in E$, $s^{(S^{-1})}(e) = s(e)$ and $r^{(S^{-1})}(e) = r(e)$. For $e_i \in \{e_1, ..., e_n\}$, $s^{(S^{-1})}(e) = w$ and $r^{(S^{-1})}(e) = v_i$.

Colloquially, $D^{(S^{-1})}$ will be the digraph D with the addition of vertex w and edges from w to any existing $v \in V$.

Definition 2.4. *Move R^{-1}: expand at an edge*
Let $D = (V, E, r, s)$ be a digraph. Let h be any edge of D and say $s(h) = v_1$ and $r(h) = v_2$. Then, $D^{(R^{-1})} = (V^{(R^{-1})}, E^{(R^{-1})}, r^{(R^{-1})}, e^{(R^{-1})})$ (D with move R^{-1} applied) is defined as such:

$$V^{(R^{-1})} = V \cup \{w\}; E^{(R^{-1})} = E \setminus \{h\} \cup \{f, g\};$$

For $e \in E \setminus \{h\}$, $s^{(S^{-1})}(e) = s(e)$ and $r^{(R^{-1})}(e) = r(e)$. For f, $s^{(R^{-1})}(f) = v_1$ and $r^{(S^{-1})}(f) = w$. For g, $s^{(R^{-1})}(g) = w$ and $r^{(S^{-1})}(g) = v_2$.

Colloquially, $D^{(R^{-1})}$ will the digraph D with one new vertex added in the middle of an edge, splitting this edge into two edges, but maintaining the original directionality.

Recently, Farsi et. al. determined which of a digraph's spectra are maintained by the each of these six moves and their inverses. These results are outlined clearly in their **Theorem 1.1**. [3, p.30]

Of particular interest to our discussion is the fact that the Hermitian adjacency spectrum is not preserved by any of the six digraph moves or their inverses [3]. This brings us to our broadest question for investigation: What can we say about the Hermitian adjacency spectra of digraphs with Morita equivalent $C*$-algebras?

3 Performing Move (S) inverse

To begin, we will look at Move $(S)^{-1}$, which is relatively simple and can be applied to any digraph at any vertex or at multiple vertices. To reiterate, the process for applying Move $(S)^{-1}$ can be summarized colloquially in two steps:

1. Add a vertex.
2. Add edges from your new vertex to any existing vertices.

3.1 A simple example

We will start with a very simple digraph:

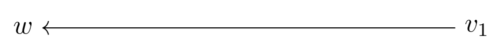


Figure 1: d_1

For this digraph, we can compute the following:

$$\mathbf{H}(d_1) = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}; \qquad Spec_{\mathbf{H}}(d_1) = \{-1, 1\}$$

Now, applying Move $(S)^{-1}$ at w, we will insert a new vertex, v_2, which emits one edge, e_2 such that $r(e_2) = w$. Doing this, we get the resulting digraph, with the indicated Hermitian adjacency matrix and spectrum:

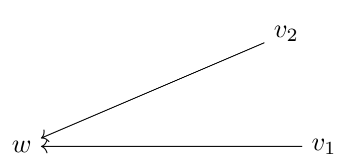


Figure 2: d_2

$$\mathbf{H}(d) = \begin{bmatrix} 0 & i & 0 \\ -i & 0 & -i \\ 0 & i & 0 \end{bmatrix}; \qquad Spec_{\mathbf{H}}(d_2) = \{-\sqrt{2}, \sqrt{2}, 0\}$$

Continuing as such, applying Move $(S)^{-1}$ repeatedly at w, we produce the following series of digraphs, each of whose Hermitian adjacency spectra are also noted.

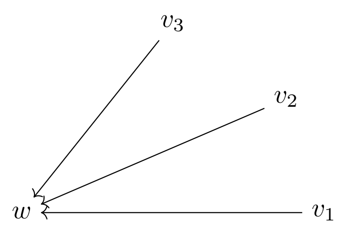


Figure 3: d_3

$$Spec_{\mathbf{H}}(d_3) = \{-\sqrt{3}, \sqrt{3}, 0, 0\}$$

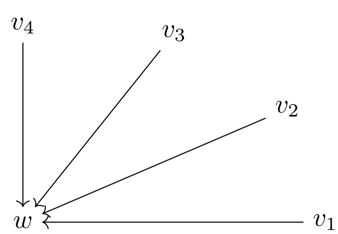


Figure 4: d_4

$$Spec_{\mathbf{H}}(d_4) = \{-2, 2, 0, 0, 0\}$$

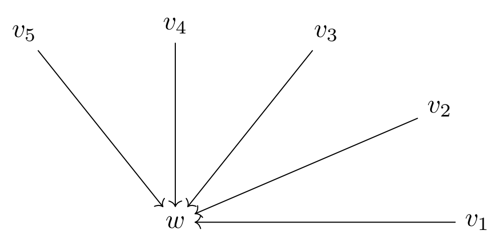


Figure 5: d_5

$$Spec_{\mathbf{H}}(d_5) = \{-\sqrt{5}, \sqrt{5}, 0, 0, 0, 0\}$$

Say we continue to follow this same pattern, adding n vertices, v_n, and edges, e_n, such that $s(e_n) = v_n$ and $r(e_n) = w$ to create digraph d_n. This will result in the following digraph:

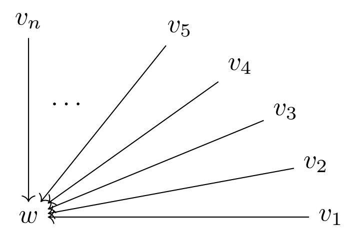


Figure 6: d_n

From observation, it appears that $Spec_{\mathbf{H}}(d_n) = \{-\sqrt{n}, \sqrt{n}, 0, 0, \ldots, 0\}$, with the multiplicity of 0 being $n-1$. However, this is merely the case of one (relatively simple) digraph. How will applying this move sequence to more complicated digraphs compare?

3.2 *fish*

To begin addressing this question, let's consider another example: the digraph *fish*.

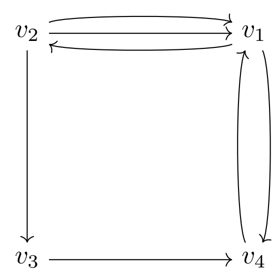


Figure 7: $fish$

In the case of *fish*,

$$\mathbf{H}(fish) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & i & 0 \\ 0 & -i & 0 & i \\ 1 & 0 & -i & 0 \end{bmatrix} \end{array} \qquad Spec_{\mathbf{H}}(fish) = \left\{-\sqrt{2}, -\sqrt{2}, \sqrt{2}, \sqrt{2}\right\}$$

Applying $(S)^{-1}$ once at v_1, we get the following:

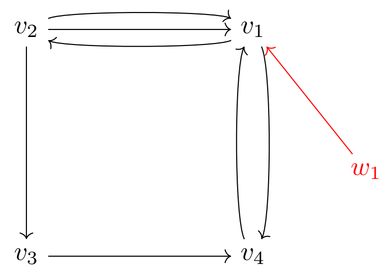


Figure 8: $fish_{11}$

$$\mathbf{H}(fish_{11}) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ w_1 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & w_1 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & -i \\ 1 & 0 & i & 0 & 0 \\ 0 & -i & 0 & i & 0 \\ 1 & 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad Spec_{\mathbf{H}}(fish_{11}) = \left\{\sqrt{3}, -\sqrt{3}, -\sqrt{2}, \sqrt{2}, 0\right\}$$

Now, we will do the same thing a few more times.

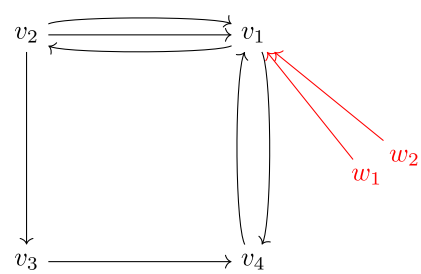


Figure 9: $fish_{12}$

$$\mathbf{H}(fish_{12}) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ w_1 \\ w_2 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & w_1 & w_2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & -i & -i \\ 1 & 0 & i & 0 & 0 & 0 \\ 0 & -i & 0 & i & 0 & 0 \\ 1 & 0 & -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad Spec_{\mathbf{H}}(fish_{12}) = \left\{2, -2, \sqrt{2}, \sqrt{2}, 0, 0\right\}$$

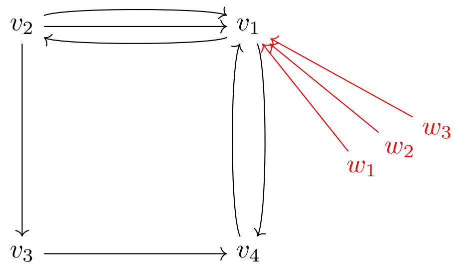


Figure 10: $fish_{13}$

$$\mathbf{H}(fish_{13}) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ w_1 \\ w_2 \\ w_3 \end{array} \begin{array}{c} \begin{array}{ccccccc} v_1 & v_2 & v_3 & v_4 & w_1 & w_2 & w_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & -i & -i & -i \\ 1 & 0 & i & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & i & 0 & 0 & 0 \\ 1 & 0 & -i & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$Spec_{\mathbf{H}}(fish_{13}) = \left\{2, -2, \sqrt{2}, \sqrt{2}, 0, 0, 0\right\}$$

The following illustrates what will happen when we perform this move at v_1 an arbitrary number of times.

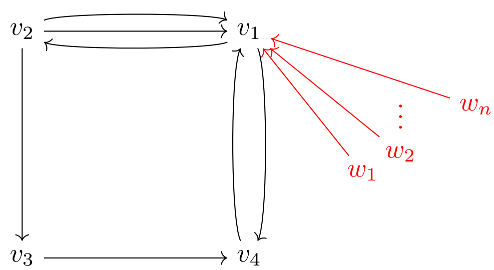


Figure 11: $fish_{1n}$

$$\mathbf{H}(fish_{1n}) = \begin{bmatrix} 0 & 1 & 0 & 1 & -i & & -i \\ 1 & 0 & i & 0 & 0 & \dots & 0 \\ 0 & -i & 0 & i & 0 & & 0 \\ 1 & 0 & -i & 0 & 0 & & 0 \\ i & 0 & 0 & 0 & 0 & & 0 \\ & \vdots & & & & \ddots & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Spec_{\mathbf{H}}(fish_{1n}) = \left\{ -\sqrt{2}, \sqrt{2}, -\sqrt{n+2}, \sqrt{n+2}, 0, 0, \dots, 0 \right\}$$

As we keep adding new vertices and edges directed at v_1, we can see that the Hermitian adjacency matrix and spectrum is following a relatively simple pattern, and is similar to the pattern for our original example. However, brief investigation into a larger, more complicated digraph will not elicit such easily noticeable patterns. Thus, how can we generalize these results?

4 Move (S) inverse and any digraph

We would like to find a way to characterize how the spectrum of *any* digraph will change when applying Move $(S)^{-1}$ repeatedly to one of its vertices. For clarity, let D_n denote the digraph D with move S^{-1} applied n times at a single vertex.

4.1 General matrix pattern

From our examples, we can see that when repeatedly applying Move S^{-1} to any digraph at a single vertex, the Hermitian adjacency matrix will follow an easily characterizable pattern. Begin by noticing that we can reorder a Hermitian adjacency matrix so that any vertex corresponds with any row and column (so long as each still corresponds with the same column as they do row). So, let's choose to organize our matrix such that the vertex which we are applying $(S)^{-1}$ to corresponds with the first row and column. Thus, we can see that, for any digraph, D, and n applications of S^{-1} (such that D_1 is $(S)^{-1}$ applied once),

$$\mathbf{H}(D_n) = \begin{bmatrix} & & & -i & & -i \\ & \mathbf{H}(D) & & 0 & \dots & 0 \\ & & & 0 & & 0 \\ i & 0 & 0 & 0 & & 0 \\ & \vdots & & & \ddots & \\ i & 0 & 0 & 0 & & 0 \end{bmatrix}$$

Here, $\mathbf{H}(D_n)$ will have size $(n + |V|) \times (n + |V|)$ matrix, where $|V|$ is the number of vertices in our original digraph.

Now, we want to see how this pattern translates into the characteristic polynomial, and in turn, to the spectral elements.

4.2 Block matrix structure

The following theorem, derived from the Schur Complement Formula, can be found in [1]. This theorem utilizes the technique of expressing a matrix in *block form*, essentially as a matrix comprised of smaller matrices.

Theorem 1 ([1, p.100])**.** Let A be an $n \times n$ matrix that can be expressed in block form as $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ with square blocks A_{11} of size $k \times k$ and A_{22} of size $(n-k) \times (n-k)$.

If A_{11}, A_{22} are invertible, then

$$Det(A) = Det(A_{11} - A_{12}A_{22}^{-1}A_{21})Det(A_{22}) = Det(A_{22} - A_{21}A_{11}^{-1}A_{12})Det(A_{11})$$

This theorem allows us to calculate the determinant of a matrix by calculating products and determinants of blocks within the matrix. We are interested in the determinant of $\mathbf{H}(D_n) - \lambda I$, as this is our characteristic polynomial, and its roots are the eigenvalues. So, decompose $\mathbf{H}(D_n) - \lambda I$ into the following blocks:

$$\mathbf{H}(D_n) - \lambda I = \left[\begin{array}{ccc|ccc} & & & -i & & -i \\ & \mathbf{H}(D) - \lambda I & & 0 & \dots & 0 \\ & & & 0 & & 0 \\ \hline i & 0 & 0 & -\lambda & & 0 \\ & \vdots & & & \ddots & \\ i & 0 & 0 & 0 & & -\lambda \end{array}\right]$$

Letting $|V|$ be the number of vertices in D and n the number of times S^{-1} is applied, and $\mathbb{C}^{s \times t}$ be the set of all complex-valued matrices with s rows and t columns,

$$A_{11} = \mathbf{H}(D) - \lambda I \in \mathbb{C}^{|V| \times |V|}; A_{12} = \begin{bmatrix} -i & & -i \\ 0 & & 0 \\ \vdots & \dots & \vdots \\ 0 & & 0 \end{bmatrix} \in \mathbb{C}^{|V| \times n};$$

$$A_{21} = \begin{bmatrix} i & 0 & \dots & 0 \\ & & \vdots & \\ i & 0 & \dots & 0 \end{bmatrix} \in \mathbb{C}^{n \times |V|}; \text{ and } A_{22} = -\lambda I \in \mathbb{C}^{n \times n}$$

Notice that this means that $A_{22}^{-1} = -\frac{1}{\lambda}I$. So, when $\lambda = 0$, that leaves this matrix undefined. However, we know the left side of our equation 1 is continuous. The right is only potentially not continuous at a finite number of points, at which it will either be zero or infinity. Since the left can never be infinity, we can conclude that the right will be zero at each of these points. That is to say, this theorem and formula will hold even when $\lambda = 0$.

Using these matrices and **Theorem 1**, we can see that

$$Det(\mathbf{H}(D_n)-\lambda I) = Det\left(\mathbf{H}(D)-\lambda I - \begin{bmatrix} -i & & -i \\ 0 & & 0 \\ \vdots & \dots & \vdots \\ 0 & & 0 \end{bmatrix}\begin{bmatrix}-\frac{1}{\lambda}I\end{bmatrix}\begin{bmatrix} i & 0 & \dots & 0 \\ & & & \vdots \\ i & 0 & \dots & 0 \end{bmatrix}\right) Det(-\lambda I)$$

Simplifying, we get that

$$Det(\mathbf{H}(D_n)-\lambda I) = Det\left(\mathbf{H}(D)-\lambda I + \begin{bmatrix} \frac{n}{\lambda} & & 0 \\ & \ddots & \\ 0 & & 0 \end{bmatrix}\right) Det(-\lambda I)$$

Setting this expression equal to zero and solving for λ to find the eigenvalues, we get that

$$Det\left(\mathbf{H}(D)-\lambda I + \begin{bmatrix} \frac{n}{\lambda} & & 0 \\ & \ddots & \\ 0 & & 0 \end{bmatrix}\right) = 0 \text{ and } Det(-\lambda I) = 0$$

The fact that $Det(-\lambda I) = 0$ means that $(-\lambda)^n = 0$, since I is the $n \times n$ identity matrix. Thus, the eigenvalue zero will have a multiplicity of at least n.

We can use the other expression to determine the non-zero elements of the Hermitian adjacency spectrum. To do this, all we must do is add $\frac{n}{\lambda}$ to the first element of $\mathbf{H}(D) - \lambda I$. Taking the determinant will then give us a characteristic polynomial in terms of n, and solving this for it's roots will tell us the non-zero values of λ for the desired digraph.

4.3 Process to determine $Spec_{\mathbf{H}(D)}$ for any D

We now have an efficient way to find the Hermitian adjacency spectrum of any digraph after $(S)^{-1}$ is applied to any one of it's vertices any number of times. The process is as follows:

1. Find the Hermitian adjacency matrix of your digraph, D.
   - Arrange $\mathbf{H}(D)$ so that the first row and column correspond with the vertex you want to apply S^{-1} to.
2. Find $\mathbf{H}(D) - \lambda I$.
3. Add $\frac{n}{\lambda}$ to the first element of $\mathbf{H}(D) - \lambda I$.
4. Take the determinant of the resulting matrix.
5. Set this characteristic polynomial equal to zero and solve for λ.

See appendix A for an example of the application of this process.

4.4 Another approach

While this is interesting, it would be ideal to be able to describe the new spectrum in terms of the old spectrum, rather than the elements of the original Hermitian adjacency matrix. To work towards this, we can split the matrix into different blocks, and use the other part of the formula noted earlier:

$$Det(A) = Det(A_{22} - A_{21}A_{11}^{-1}A_{12})Det(A_{11})$$

This time, adjust $\mathbf{H}(D)$ so that its last row and column correspond with the vertex at which we are performing S^{-1}. For simplicity, we will call this vertex v. Assume that there are no loops at v, meaning that the (v, v) entry of $\mathbf{H}(D)$ will be zero. Let $\mathbf{H}_p$ be $\mathbf{H}(D)$ minus the row and column associated with v. Let $\mathbf{v}_r$ be the row vector associated with v minus the (v, v) entry, and let $\mathbf{v}_c$ be the column vector associated with v minus the (v, v) entry. This is illustrated below:

$$\mathbf{H}(D) = \begin{bmatrix} & & & | \\ & \mathbf{H}_p & & \mathbf{v}_c \\ & & & | \\ — & \mathbf{v}_r & — & 0 \end{bmatrix}$$

Now, we can break $\mathbf{H}(D_n) - \lambda I$ into the following blocks:

$$\mathbf{H}(D_n) - \lambda I = \left[\begin{array}{ccc|cccc} & & & | & 0 & & 0 \\ & \mathbf{H}_p - \lambda I & & \mathbf{v}_c & 0 & \dots & 0 \\ & & & | & 0 & & 0 \\ \hline — & \mathbf{v}_r & — & -\lambda & -i & ... & -i \\ 0 & 0 & 0 & i & -\lambda & & 0 \\ & \vdots & & \vdots & & \ddots & 0 \\ 0 & 0 & 0 & i & 0 & 0 & -\lambda \end{array}\right]$$

Applying Theorem 1, we get that

$$Det(\mathbf{H}(D_{Sn})) = Det(\mathbf{H}_p - \lambda I) \times Det \begin{bmatrix} -\lambda - * & -i & ... & -i \\ i & -\lambda & & 0 \\ \vdots & & \ddots & 0 \\ i & 0 & 0 & -\lambda \end{bmatrix}$$

Such that $* = \mathbf{v}_r(\mathbf{H}_p - \lambda I)^{-1}\mathbf{v}_c$. As with our other block decomposition of $\mathbf{H}(D) - \lambda I$, we can see that only the top left element is changing.

Thus far, $Det \begin{bmatrix} -\lambda - * & -i & ... & -i \\ i & -\lambda & & 0 \\ \vdots & & \ddots & 0 \\ i & 0 & 0 & -\lambda \end{bmatrix} = (-1)^n \left(n + \lambda^{1+n} + \frac{x}{Det(\mathbf{H}_p - \lambda I)} + \frac{y}{Det(\mathbf{H}_p - \lambda I)}\right).$

Currently x and y are unknown. However, this formula implies that the eigenvalues of $\mathbf{H}_p$ can be replaced because of their presence in the denominator of the determinant of the other matrix. However, sometimes they will not be, if x, y contain expressions shared with $Det(\mathbf{H}_p - \lambda I)$. Ultimately, there is potential here to derive information from this formula, but more research must be done.

5 Move (S) inverse at different vertices

Now that we've found a way to determine characteristic polynomials and spectral elements for digraphs with $(S)^{-1}$ applied multiple times at one vertex, we will look at what happens when we perform this move sequence at different vertices of a single digraph, one at a time. Will each vertex elicit the same pattern?

To investigate this, we will look at the following digraph, *spider*:

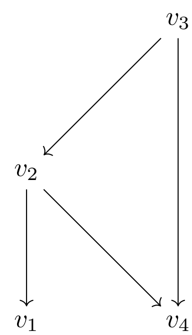


Figure 12: *spider*

We want to investigate what patterns arise when we apply Move $(S)^{-1}$ repeatedly to each of the vertices. To do so, using the process outlines in section 4.3, we will find the resulting characteristic polynomials along with the spectral elements. Comparing these will allow us to see the difference in the effect of this move sequence at each vertex more clearly. For this example, let $spider_{mn}$ or s_{mn} be the digraph with $(S)^{-1}$ applied n times to vertex m.

5.1 Applying Move (S) inverse at v_1

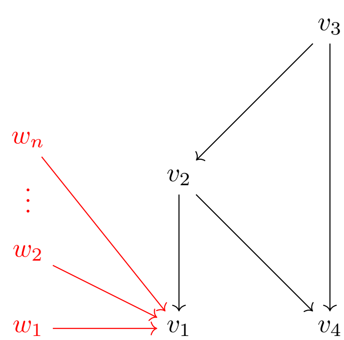


Figure 13: $spider_{1n}$

The characteristic polynomial of $spider_{1n}$ will be: $p(s_{1n}) = (-\lambda)^n \left((3n+1) - \lambda^2(n+4) + \lambda^4\right)$

Solving for λ, we get that the four non-zero elements of the spectrum will be:

$$\left\{\lambda \to -\frac{\sqrt{-\sqrt{n^2-4n+12}+n+4}}{\sqrt{2}}\right\}, \left\{\lambda \to \frac{\sqrt{-\sqrt{n^2-4n+12}+n+4}}{\sqrt{2}}\right\},$$

$$\left\{\lambda \to -\frac{\sqrt{\sqrt{n^2-4n+12}+n+4}}{\sqrt{2}}\right\}, \left\{\lambda \to \frac{\sqrt{\sqrt{n^2-4n+12}+n+4}}{\sqrt{2}}\right\}$$

5.2 Applying Move (S) inverse at v_2

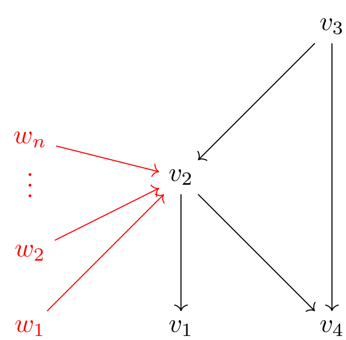


Figure 14: *spider*$_{2n}$

The characteristic polynomial of *spider*$_{2n}$ will be: $p(s_{2n}) = (-\lambda)^n \left((n+1) - \lambda^2(n+4) + \lambda^4\right)$

And the four non-zero element of $Spec_{\mathbf{H}}(s_{2n})$ are:

$$\left\{\lambda \to -\frac{\sqrt{-\sqrt{n^2+4n+12}+n+4}}{\sqrt{2}}\right\}, \left\{\lambda \to \frac{\sqrt{-\sqrt{n^2+4n+12}+n+4}}{\sqrt{2}}\right\},$$

$$\left\{\lambda \to -\frac{\sqrt{\sqrt{n^2+4n+12}+n+4}}{\sqrt{2}}\right\}, \left\{\lambda \to \frac{\sqrt{\sqrt{n^2+4n+12}+n+4}}{\sqrt{2}}\right\}$$

5.3 Applying Move (S) inverse at v_3

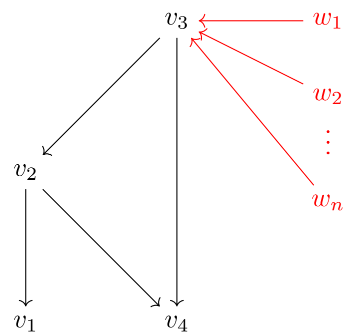


Figure 15: $spider_{3n}$

The characteristic polynomial of $spider_{3n}$ will be: $p(s_{3n}) = (-\lambda)^n \left((2n+1) - \lambda^2(n+4) + \lambda^4\right)$

And the four non-zero element of $Spec_{\mathbf{H}}(s_{3n})$ are:

$$\left\{\lambda \to -\frac{\sqrt{-\sqrt{n^2+12}+n+4}}{\sqrt{2}}\right\}, \left\{\lambda \to \frac{\sqrt{-\sqrt{n^2+12}+n+4}}{\sqrt{2}}\right\},$$

$$\left\{\lambda \to -\frac{\sqrt{\sqrt{n^2+12}+n+4}}{\sqrt{2}}\right\}, \left\{\lambda \to \frac{\sqrt{\sqrt{n^2+12}+n+4}}{\sqrt{2}}\right\}$$

5.4 Applying Move (S) inverse at v_4

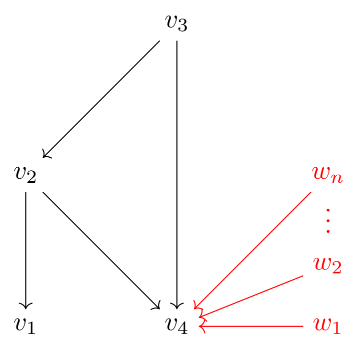


Figure 16: $spider_{4n}$

The characteristic polynomial of $spider_{4n}$ will be: $p(s_{4n}) = (-\lambda)^n \left((2n+1) - \lambda^2(n+4) + \lambda^4\right)$

And the four non-zero element of $Spec_{\mathbf{H}}(s_{4n})$ are:

$$\left\{\lambda \to -\frac{\sqrt{-\sqrt{n^2+12}+n+4}}{\sqrt{2}}\right\}, \left\{\lambda \to \frac{\sqrt{-\sqrt{n^2+12}+n+4}}{\sqrt{2}}\right\},$$

$$\left\{\lambda \to -\frac{\sqrt{\sqrt{n^2+12}+n+4}}{\sqrt{2}}\right\}, \left\{\lambda \to \frac{\sqrt{\sqrt{n^2+12}+n+4}}{\sqrt{2}}\right\}$$

5.5 Comparison and conjecture: underlying degree

The following are the resulting formulae for the characteristic polynomial of *spider* with $(S)^{-1}$ applied n times at each of the four vertices:

$$p(s_{1n}) = (-\lambda)^n \left(3n+1 - \lambda^2(n+4) + \lambda^4\right)$$

$$p(s_{2n}) = (-\lambda)^n \left(n+1 - \lambda^2(n+4) + \lambda^4\right)$$

$$p(s_{3n}) = (-\lambda)^n \left(2n+1 - \lambda^2(n+4) + \lambda^4\right)$$

$$p(s_{4n}) = (-\lambda)^n \left(2n+1 - \lambda^2(n+4) + \lambda^4\right)$$

As shown, the only variation between each characteristic polynomial is the coefficient of λ^n. Additionally, vertices v_3 and v_4 follow the same pattern. To think about why this might be, let's look at our original digraph again:

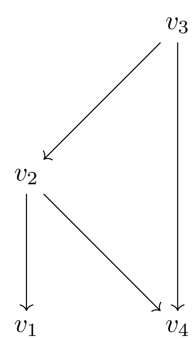


Figure 17: *spider*

Visually, we can see that both v_3 and v_4 are adjacent to two other vertices, whereas v_2 is adjacent to three and v_1 to one. However, note that, v_3 has two edges pointing *away* from it, while v_4 has two edges pointing *towards* it. In order to consider adjacency and degree while disregarding the directionality of edges, we can look at a digraph's underlying graph and a vertex's underlying degree. Recall:

- *underlying digraph*: the graph obtained by making every edge undirected and then removing any parallel edges

- *underlying degree of a vertex*: the number of edges a single vertex touches in a digraph's underlying graph

The underlying graph of *spider* with degrees of each vertex noted in blue is show below.

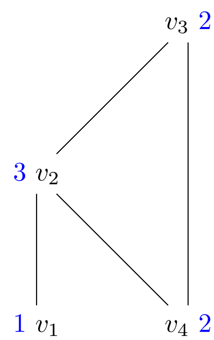


Figure 18: *spider*

Conjecture. Within a single digraph, D two vertices who have the same *underlying degree* will follow the same spectral pattern when $(S)^{-1}$ is applied to them repeatedly.

Thus far, this conjecture has remained true for every digraph it has been tested on, including *fish*, whose underlying graph is shown below. Every vertex of the underlying graph of *fish* has the same degree and follows the same pattern when $(S)^{-1}$ is applied to it.

6 Spectral radius

Switching gears slightly, we will now look into spectral radius and the extent to which the Hermitian adjacency spectra can change through move sequences. When considering a matrix, $\mathbf{A}$, as inducing a transformation such that $T(\mathbf{x}) = \mathbf{Ax}$, the eigenvalues of A represent the scales to which different vectors can be multiplied, but remain with the same direction. Thus, the spectral radius is the largest scale by which you can multiply an element through this transformation and maintain its direction. We would like to determine how spectral radius behaves in general, and particularly, if it can tend towards infinity through these digraph moves.

Recall these definitions from earlier:

- The *spectral radius* of a matrix is the maximum of the absolute values of its set of eigenvalues.
- The *maximum degree of an underlying graph* is the largest degree of a single vertex in the underlying graph of a digraph.

By Guo and Mohar's **Theorem 5.1** [4, p.10], the spectral radius of $\mathbf{H}(D)$ is bounded above by the maximum degree of the underlying graph. Recall *fish* from earlier. The following is the underlying graph of *fish*.

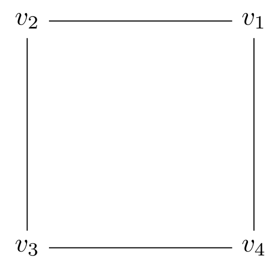


Figure 19: The Underlying Graph of *fish*

The maximum degree of this underlying graph is $2 \geq \sqrt{2} = max(Spec_{\mathbf{H}}(fish))$, which is consistent with **Theorem 5.1** from [4, p.10].

6.1 (S) inverse and spectral radius

Applying Move $(S)^{-1}$ repeatedly to the same vertex continuously increases the maximum degree of the underlying graph. Additionally, this move increases the spectral radius of the digraph to which it is applied. This fact can be proven using Guo and Mohar's **Theorem 4.1** [4, p.8], which assures that eigenvalues of D_{Sn} will interlace between those of $D_{S(n+1)}$. (Here, D_{Sn} is the digraph with $(S)^{-1}$ applied n times to one vertex.)

From the outlined examples, it appears that spectral radius increases around the order of $\sqrt{n}$. That is, $(S)^{-1}$ applied n times at one vertex will yield a spectral radius around $\sqrt{n}$. This does mean that as n approaches infinity, so will the spectral radius. However, this is merely a conjecture.

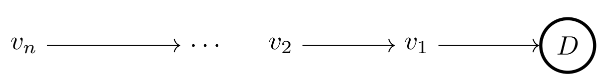
6.2 Bounded spectral radius

We can also consider move sequences which raise the maximum degree of the underlying graph only on their first application, or not at all. For these, we know that the spectral radius will be bounded as we continue applying the move sequence. This means that the spectral radius must converge to some number. But what will it converge to? Will it be the maximum degree of the underlying digraph (the upper bound on the spectral radius)?

Two such move sequences are: Move $(S)^{-1}$ applied to itself, and Move $(R)^{-1}$ applied to a single edge, explained below.

6.2.1 Move (S) inverse applied to itself

In this move sequence, we will apply $(S)^{-1}$ to any existing vertex, then apply $(S)^{-1}$ to the most recently added vertex with each iteration. That looks like this:



With examples like this, the spectral radius does not seem to approach its upper bound. Rather, it approaches some unknown number very quickly, in ten iterations or so, then increases in very small increments from there. To determine the specific number the spectral radius converges to with this move sequence will require more research.

6.2.2 Move (R) inverse applied to a single edge

Recall that Move (R) is to reduce at a regular vertex, or to remove some vertex, v, such that every edge whose range was v now has the range of the one edge that had v as a source. It's precise definition is given earlier, but the effect of $(R)^{-1}$ is to essentially insert a vertex into an edge. We can do this to any edge, any number of times, and it will not raise the maximum degree of the underlying graph. To illustrate, here is *spider* with $(R)^{-1}$ applied to the edge from v_3 to v_2:

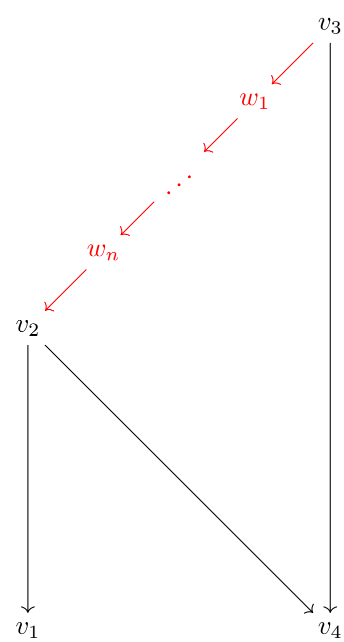


Figure 20: $spider_{Rn}$

In terms of spectral radius, examples have shown similar results to those seen with $(S)^{-1}$ applied to itself. The radius converges to an unknown number that is not the upper bound on the spectral radius as outlined in **Theorem 5.1** [4]. However, with this example, in the first few iterations of the move, the spectral radius sometimes oscillates, rather than purely growing. Again, more research will need to be done to determine particularly what the spectral radius converges to in this case.

6.3 Forcing Spectral Radii to infinity

Theorem 5.1 in [4] also outlines the specific instances in which the radius of a digraph's Hermitian adjacency spectrum will be equal to its upper bound (the maximum degree of its underlying graph). Hypothetically, if we found a sequence of moves that can be done indefinitely and maintain all requirements in this theorem, we would be able to ensure the spectral radius tends towards infinity as we continue performing these moves.

For a quick summary, for the spectral radius of a digraph D to equal its upper bound, there must be a partition of $V(D)$, the vertex set of D, that matches one of the two following illustrations.

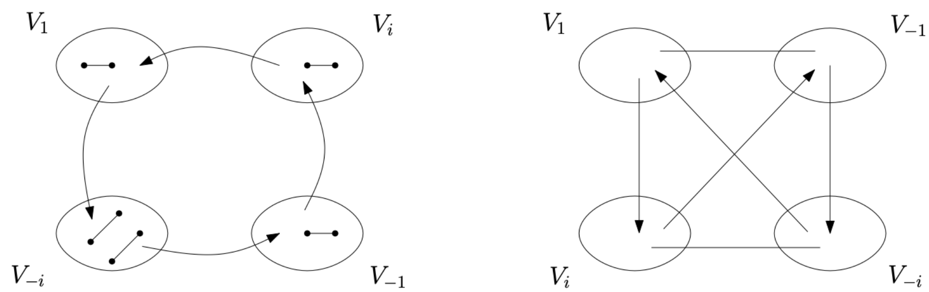


Figure 21: Partition of $V(D)$

In addition, the underlying digraph of D must be $r - regular$, meaning every $v \in V(D)$ must have the same underlying degree.

In attempting to find a move sequence that maintains these qualities of our digraph, Move (O) was used. Move (O): outsplit at a non-sink can be summarized as follows

- Choose a vertex v
- Delete v and replace it with n new vertices, keeping their range the same
- Split the set of edges which originally pointed away from v among the new vertices
- For every edge that originally pointed towards v, add n new edges pointing from the original source to each new vertex

This move can successfully maintain the partitions required, as shown in the digraphs below. Starting with the digraph D, we will apply Move (O) at v_1, splitting it into w_1 and w_2.

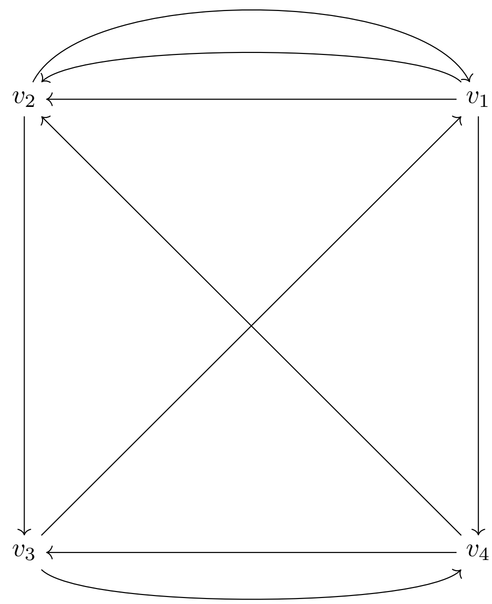


Figure 22: D

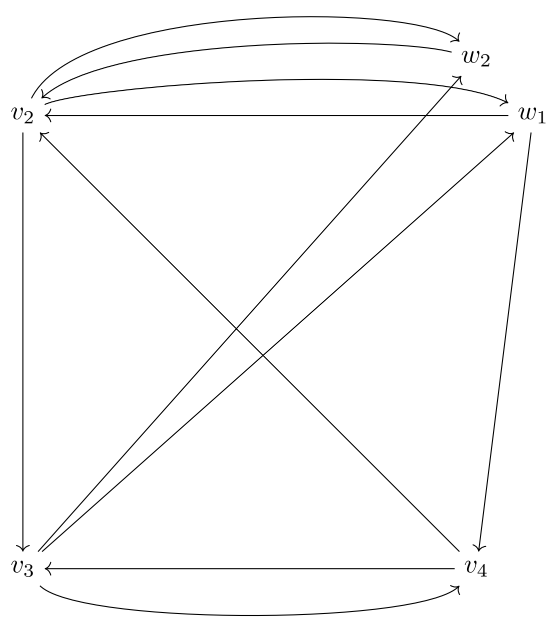


Figure 23: D with Move (O) applied at v_1

However, notice that in our resulting digraph, the underlying degrees of each vertex are not equal. Thus, we have not maintained all the requirements of the theorem. Currently, it appears that this is as successful as the digraph moves can be at maintaining all of these stipulations. While the partitions can be maintained, the underlying digraph does not remain $r-$regular

7 (R) inverse matrix pattern

While we cannot presently determine what the spectral radius converges to as we repeatedly apply Move $(R)^{-1}$, can we characterize how spectral elements change as we did with $(S)^{-1}$ applied to one vertex?

Because move $(R)^{-1}$ involves adjusting an existing edge of the digraph, it does not maintain the original Hermitian adjacency matrix as a block within the newly-created matrices. Thus, using the Schur complement formula [1] to characterize its change is more complicated.

Recalling the original Hermitian adjacency matrix of *spider*, let's compare it to that of $spider_{R3}$, or *spider* with $(R)^{-1}$ applied three times, as shown below.

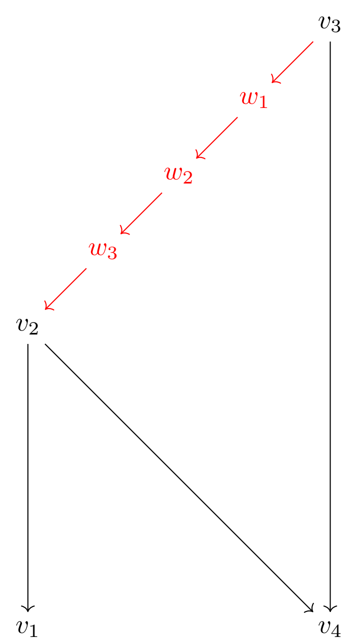


Figure 24: $spider_{R3}$

$$\mathbf{H}(spider) = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & -i & i \\ 0 & i & 0 & i \\ 0 & -i & -i & 0 \end{pmatrix}$$

$$\mathbf{H}(spider_{R3}) = \begin{pmatrix} 0 & -i & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & i & 0 & 0 & -i \\ 0 & 0 & 0 & i & i & 0 & 0 \\ 0 & -i & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & -i & 0 & i \\ 0 & i & 0 & 0 & 0 & -i & 0 \end{pmatrix}$$

Notice that, highlighted in red, the $(2,3)$ and $(3,2)$ entries of the original matrix are moved to the $(2,5)$ and $(5,2)$ entries, respectively. This corresponds with performing the move at the edge between v_3 and v_2.

In general, for this example, this particular matrix follows this pattern:

$$\mathbf{H}(spider_{Rn}) = \begin{pmatrix} 0 & -i & 0 & 0 & 0 & 0 & & 0 & 0 & 0 \\ i & 0 & 0 & i & 0 & 0 & & 0 & 0 & -i \\ 0 & 0 & 0 & i & i & 0 & & 0 & 0 & 0 \\ 0 & -i & -i & 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & i & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 & & 0 & 0 & 0 \\ & & & & & & \ddots & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & i \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \end{pmatrix}$$

For other digraphs, the Hermitian adjacency matrix will follow a similar pattern: Splitting an edge from v_x to v_y will move the (x,y) and (y,x) entries outside of the upper left block. The (n,y) and (y,n) entries will be i and $-i$ respectively, representing the edge from the n^{th} new vertex to v_y. Then, the lower right block will follow the pattern shown of i's above the main diagonal and $-i$'s below. Further research could look into translating this matrix pattern into a spectral pattern.

8 Conclusion

Through our preliminary exploration of digraph moves and the Hermitian adjacency spectrum, we have found numerous patterns. Particularly, we have looked at Move $(S)^{-1}$ in depth. This move, which can be applied to any digraph, D, any number of times, appears to send the spectral radius to infinity. It also elicits a pattern in the Hermitian adjacency matrix which allows us to use this original $\mathbf{H}(D)$ to describe the spectra of D with $(S)^{-1}$ applied any number of times to one vertex. In applying this move at different vertices of D, we also observe that underlying degree of a vertex appears to play a role in the patterns followed by spectral elements.

Notably, $(S)^{-1}$ can be applied a number of different ways. Further study could use these techniques to determine how it changes when we perform this move at a random selection of vertices at different times, rather than the same on repeatedly.

Additionally, Move $(R)^{-1}$ can be applied to any digraph – provided that digraph has at least one edge. Move $(R)^{-1}$ also elicits a consistent, characterizable pattern in the Hermitian adjacency matrix of a digraph, which opens the door for further investigation.

While no distinctly proven claims can yet be said about the spectral radius of a digraph's Hermitian adjacency spectra at this time, there is suggestion that this radius can tend towards infinity through these moves (particularly through $(S)^{-1}$) – leaving us within the same Morita equivalence class. With the use of results from [4], proving this claim appears possible.

A *Jellyfish*

To illustrate the process outlined in sect. 4.3, here is an example. Blue edges and vertices represent those which will be added through applications of Move $(S)^{-1}$. Note that $\mathbf{H}(D)$ is arranged so that the first row and column correspond with the vertex in the center of the graph, at which we are applying $(S)^{-1}$.

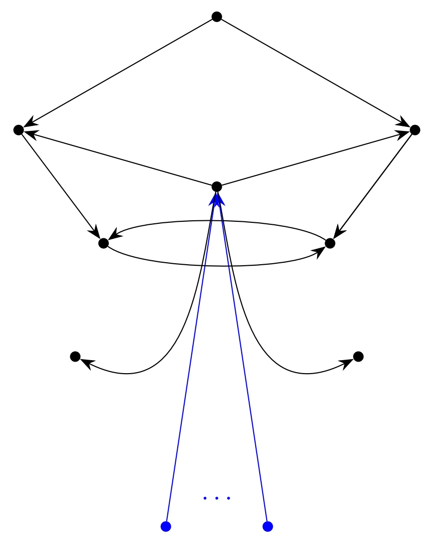


Figure 25: $jellyfish_n$

$$\mathbf{H}(jellyfish) = \begin{bmatrix} 0 & 0 & -i & 0 & -i & 0 & i & i \\ 0 & 0 & -i & 0 & 0 & 1 & 0 & 0 \\ i & i & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & i & 0 & 0 & 0 \\ i & 0 & 0 & -i & 0 & i & 0 & 0 \\ 0 & 1 & 0 & 0 & -i & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, we will find $\mathbf{H}(D) = \lambda I$, and subtract n/λ from its first entry. This gives us the resulting matrix.

$$\begin{bmatrix} -\lambda+\frac{n}{\lambda} & 0 & -i & 0 & -i & 0 & i & i \\ 0 & -\lambda & -i & 0 & 0 & 1 & 0 & 0 \\ i & i & -\lambda & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & -\lambda & i & 0 & 0 & 0 \\ i & 0 & 0 & -i & -\lambda & i & 0 & 0 \\ 0 & 1 & 0 & 0 & -i & -\lambda & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 & -\lambda & 0 \\ -i & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda \end{bmatrix}$$

The determinant of this matrix is the following expression:

$$\frac{\lambda^9 - 9\lambda^7 + 19\lambda^5 - 4\lambda^4 - 10\lambda^3 + 4\lambda^2 - \lambda^7 n + 5\lambda^5 n - 5\lambda^3 n + 2\lambda^2 n}{\lambda}$$

Solving for λ, we get three constant eigenvalues: $0, \frac{1}{2}\left(-\sqrt{5}-1\right), \frac{1}{2}\left(\sqrt{5}-1\right)$. The other nonzero eigenvalues will be the solutions to $\lambda^5 - \lambda^4 + \lambda^3(-n-7) + \lambda^2(n+6) + \lambda(3n+6) - 2n - 4 = 0$. To illustrate further, here are the nonzero elements of the spectra for cases where $(S)^{-1}$ is applied once, five times, 25 times, and 100 times:

$$Spec_{\mathbf{H}}(jellyfish_1) = \{-2.54056, -1.61803, -1.09369, 0.583819, 0.618034, 1.39056, 2.65987\}$$

$$Spec_{\mathbf{H}}(jellyfish_5) = \{-3.12808, -1.61803, -1.36212, 0.603996, 0.618034, 1.71597, 3.17023\}$$

$$Spec_{\mathbf{H}}(jellyfish_{25}) = \{-5.40515, -1.61803, -1.5531, 0.614499, 0.618034, 1.93561, 5.40814\}$$

$$Spec_{\mathbf{H}}(jellyfish_{100}) = \{-10.2009, -1.61803, -1.60125, 0.617106, 0.618034, 1.98395, 10.2011\}$$

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[1] Harry Dym, *Linear algebra in action*, American Mathematical Association, 2013.

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[3] Carla Farsi, Emily Proctor, and Christopher Seaton, *The spectra of digraphs with morita equivalent c∗-algebras*, Linear Algebra and its Applications **655** (2022), 28–64.

[4] Krystal Guo and Bojan Mohar, *Hermitian adjacency matrix of digraphs and mixed graphs*, 2015.